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GEOMETRIZATION AND VISUAL REPRESENTATION OF STEREOMETRIC PROBLEMS

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Geometrization and a visual representation of stereometric problems

The article proposes to teach Euclidean geometry on the basis of its natural constructivism in the pedagogical university. The author notes that the process needs to be systematically organized, what would each time to solve different characters and different levels of complexity of the task graphic (or semi-graphical) methods. Such visual positional and metric graphical representations should encourage the formation of professional competencies and motivate the teaching and educational interest. In the article, while not ruled out the computational component of the problem, and active, exploratory approach to the use of regular concepts and facts, the ability to retrieve them from the appropriate memory should be the subject of education benchmarks creative development of the means of geometry. Proven methods of teaching and learning will provide students with the formation of dynamic stereotype representations and imaginations, visual-imaginative and logical thinking, and hence the accumulation of solid, thorough knowledge of geometry in general and elementary geometry, in particular, the skills to use them in the life and work of chosen specialty. This methodical approach will significantly increase the level of scientific and methodological training of future teachers of mathematics; will have a significant impact on the formation of positive human qualities. In the article demonstrates the possibility of saturation of the geometric content of regular tasks on the calculation. In the context of the constructive approach is promoted as a universal method of internal projection. Prospects of the study are intended to substantiate scientifically reception geometrization and visible visualization tasks, organize and structure the problems in order to bring an innovative approach to quality mastering the discipline of "Geometry".

Key words: geometrization, visual representation, graphics and graphic-analytical methods, internal projection imaging.

A contradiction emerged between legitimate public documents about the level of requirements for the preparation of mathematics teachers and the available content, forms and methods of teaching geometric subjects, which is determined by the fact of insufficient development of didactics basics and firstly by the lack of calibrated, reliable theoretical-methodological study of Euclidean geometry. In pedagogical universities the importance of innovative pedagogical technologies based on constructive approach has not yet been emphasized.

Traditionally, a formal logical approach is the foundation of teaching and learning. Pictorial modeling as a method of development of eye-mindedness, logical thinking and imaginary mastery of the first subject patterns are neglected.

Positional and metric problems on binary isomorphic models of bodies and their combinations, graphics and graphic-analytical methods for solving problems are also not worried about. It seems that the term "visualization", only in theoretical terms, is more interesting for teachers and psychologists rather than in practical future for professional geometries.

In the formulation of the problem it is meant that students are already familiar with elementary course. Now the task of detailed visualization and their knowledge structuring has not yet been established by, involving diverse geometric system solution offers on a constructive basis is set , and as a result by efficient and deep rethinking and assimilation of the first science on professional level.

Visibility as a fundamental principle of didactics was first formulated by J. A. Comenius. He believed that "with not a verbal explanation of things, but with their real obseravtion" any learning should begin. The views of J. A. Comenius were supported and developed by great teachers of the past J. H. Pestalozzi, K. D. Ushinskiy. Particularly, K. D. Ushinskiy supposed that visual presentation of facts: "is learning that is based not on abstract ideas and words, but on definite images directly perceived by the child. ... This course of learning , from concrete to abstract, from idea to thought is so natural and based on so transparent mental laws that only the one can reject its necessity, who rejects the necessity to consider the demands of human nature in studies generally and particularly child's"[1 , p. 265 – 266]. It is difficult to overestimate the points stated above, only the indifference to realize its essence is left.

Psychological studies on the use of various means of clarity were conducted by L. V. Zankov, L. I. Mendelshtam, I. N. Soloviev, N. A. Usov, L. Friedman,

J. I. Schiff, etc. According to L.V. Zankov, visibility and education provide the widespread use of visual sensations, perceptions , images and constant reliance on the evidence of the senses, which is achieved due to direct contact with reality [2]. L. M. Friedman, in a modern interpretation of the principle of didactic presentation, emphasized its role in improving the quality of learning and skills in order to improve the management work of teachers . Summarizing their own scientific research, professor L. M. Friedman L. M. emphasizes that "visibility is understanding and activity"[3, p. 60]. N. A. Usova qualifies visibility as a category of psychology and didactics, which provides a link between the concrete and the abstract, which contributes to the development of thinking and in many cases is its reliable support [4]. In "The Pedagogical Dictionary" a visibility in education is defined as "a didactic principle according to which learning is based on specific images directly perceived by students" [5, p. 727].

Geometry stands out from mathematical sciences with its exceptional esthetic appeal and visualized beauty. It is the first science [6], which for a long time was considered as a superior school of wisdom. Learning science of “Geometry" develops and perfects one’s thinking. There is a historical fact that above the entrance of the Academy, founded by the ancient Greek geometer and philosopher Plato, there was engraved: "The one who is ignorant in geometry, please do not enter!”.

Geometry was brightly idealized by the professor. A. D. Aleksandrov. He said: "The peculiarity of elementary geometry, among other branches of mathematics is that it unites strict logic with a visual perception, logical analysis with a holistic and synthetic perception of an object. We can say that in substance, geometry is nothing else but an organic combination of strict logic with visual perception: visual representation permeated and organized by strict logic, and logic awakened by visual representation .Where these components do not exist, the true geometry does not exist too" [7, p. 282 – 283].

Qualified geometrization, measurable, appropriate illustration of learning aims to promote understanding and synthesis of the material. The right hemisphere of the brain is actively connected to logic of knowledge of how things work, because only it is responsible for sensory, visual-imaginative sphere of consciousness, dimensionally-imaginative type of activity. The culture if dimensional and logical thinking, forming of research skills using the means of integral geometry are inherent from capture technologies of mental work with its imaginary objects. This is known in visual solution of calculation problems, proof and construction using graphic and graphic-analytical methods.

We aim to show examples of the possibilities for the teacher to provide a purely geometric meaning for common computing problems. Visual-imaginative modeling- dynamic representation of the algorithm of step by step actions, activity of each dimensional operations surface rendering (for instance, using the method of internal projection), personal and strictly justified removal from its own memory "just that" laws and facts on the way to the result will add beauty to this miracle science, interest in the subject of education, convince in its natural and practical life giving.

Problem 1. *In the right square pyramid an angle between neighboring lateral faces is equal to 2α. Find the lateral surface of the pyramid if the content of the diagonal section is equal to S.*

Students in seeking a solution to the problem (fig.1) might think like this: If *BKD* is a linear angle of dihedral angle at the edge of the *SC;* the point *K* which belongs to the edge of the *SC*, gets anywhere on the drawing-picture. Let's find the points *O* and *K*. It is easy to prove that *BK DK*, and therefore *OK* – a median is of *BKD* triangle is its bisector and altitude. So, $\angle OKD = \alpha$. We note that $S_{SAC} = 2S_{SOC} = SC \cdot OK$, and $S_b = 4S_{SCD} = 2SC \cdot DK$, so $\frac{S_{SAC}}{S_b} = \frac{1OK}{2DK}$. From the right triangle *ODK we* have: $\frac{OK}{KD} = \cos\alpha$ and therefore,$S_b = \frac{2S}{\cos\alpha}$.

On the other hand, if we imagine an axial section of the pyramid (*SAC*) as the projection content of internal orthogonal projection in the direction $D \to O$, then triangle *SOC* is the projection of the triangle *SDC*. Taking into account the well-known theorem about the orthogonal projection content of polygon, we obtain: $S_{SOC} = S_{SDC} \cdot \cos\alpha$. So, that is why $S_{SDC} = \frac{S}{2\cos\alpha}$ and $S_b = \frac{2S}{\cos\alpha}$.

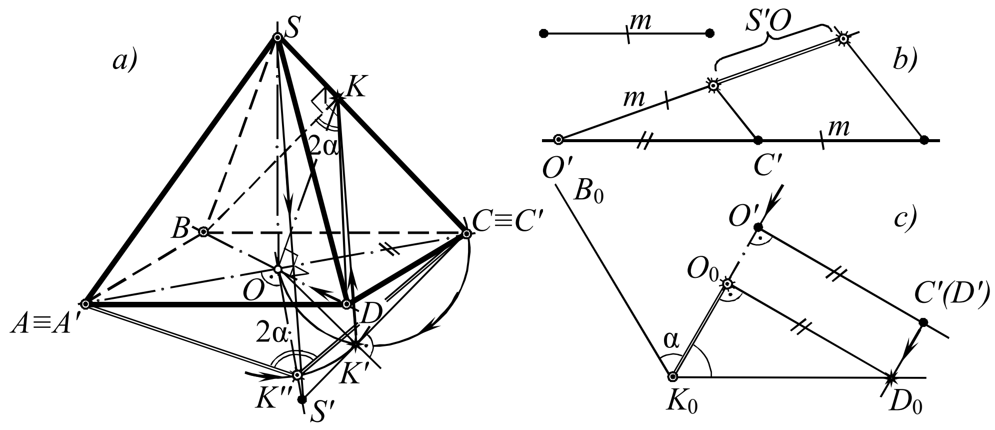

Fig. 1

Perhaps this is the shortest way to formal result, more original, but also geometrically attractive with apparent dynamic projection operation, implemented inside the body by the subject of education.

In order to represent the point *K* on the edge of the pyramid *SC* as the drawing model we need to define the content of the diagonal section S in the picture, or with the same degree measure of the angle *B'K'D'*, which is conditionally equal to 2α.

In the first case $S = m^2 = O'C' \cdot S'O' \Rightarrow S'O' = \frac{m \cdot m}{O'C'}$, where m is a given interval. Imaginedmoving the pyramid in the space so that the diagonal of basics *A'C'* "lay" on the content picture (*OS* = *O'S')* and, in addition, doing the building of the interval *S'O'* in the free space of the picture as the fourth interval proportional to *m*, *m* and *O'C'*(fig. 1b), we combine the triangle *S'O'C'* with the content of the board (exercise book).

Two final steps of building($O'K' \perp S'C'$ and $K'K \parallel S'S$) lead to the result. It is obvious that the point *K* on the edge of *SC* was found with graphic-analytical method. The angle is displayed in life-size with the angle *A'K'C'* which isequal to the angle *B'K'D'*.

If you set a degree measure as $(B_0K_0D_0)$ of the angle *B'K'D'*, then the true length of the interval *O'K'* can be found if we construct separately (fig. 1c) a right triangle $K_0O_0D_0$ behind the acute angle $\alpha = \frac{1}{2}\angle B_0K_0D_0$and cathetus $O_0D_0 = O'D'$, which is equal to the original angled triangle *K'O'D'*, because *O'D'* = *O'C',* and

$O'K' = O_0K_0$ ($O' = O$). Since the triangle *O'K'C'* is really square too, we should look for the point *K* in the intersection of two circles, one of which we set off on the interval *O'C'* as on the diameter, and the second - centered at the point *O'* and radius equal to the length of the constructed interval *O'K'*. Connecting points *K'* and *C'* with a half-line we fix the point *S'* on perpendicular up to *O'C'* at point *O'*. Then we finish the building with the graphical method mentioned above.

Problem 2. *The height of the right triangular pyramid is equal to H. Find its total surface if the content drawn across the apex of the pyramid base is perpendicular to the opposite side of the edge, and make an angle 30 ° with the base content.*

In this case (fig. 2), it is convenient to choose the base content of the pyramid $\Delta(ABC)$ as the projection content of internal orthogonal projection in the direction $S \to O$, because its lateral faces *SAB, SBC* and *SAC* will have equal triangles *AOB*, *BOC* and *AOC* as their projections. S_o, $S_n = S_o + S_b = 3(S_{AOB} + S_{SAB})$ (*).

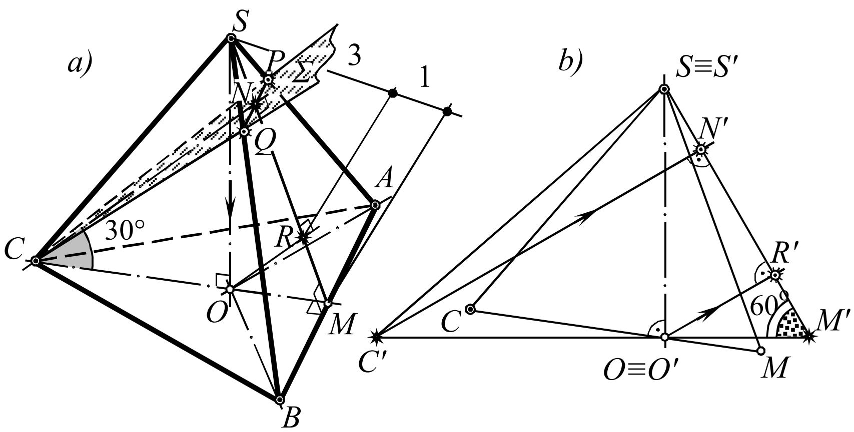

Fig. 2

Now let's talk about professional completion of the drawing-picture. We will start with setting off the interval *SM* – apothem of the face *AB*. The edge *AB*, universal for faces *SAB* and *ABC* and is perpendicular to two right lines *SM* and *CM*, which actually determine the intersection the content$\Lambda(SMC)$ of the pyramid'saxial cross section. Thus, the content $\Lambda(SMC)$ is perpendicular to the face *SAB* and the face *ABC*. So, two contents are relatively perpendicular if each of them goes through the right line which is perpendicular to another content.

Therefore, the point *N* will be thebasis of perpendicular *CN* to the face *SAB*, which encompass the content $\Lambda(SMC)$, which of course is taken elsewhere on the apothem *SM* of this face. In turn, through the point *C* (right line *CN*) we can hold a cluster of contents perpendicular to the face *SAB*. Obviously that in this cluster we should choose such a content $\Sigma(CPQ)$ from the problem situation, which would be parallel to $AB(PQ \parallel AB)$, and therefore is perpendicular to the content $\Lambda(SMC)$. Then the axial content of the section $\Lambda(SMC)$ at the intersection with its perpendicular contents $\Sigma(CPQ)$ and $\Lambda(ABC)$ select in the drawing a straight angle ($\angle NCM = 30^{\circ}$ according to the problem situation) which will be used to measure the dihedral angle formed by these contents.

Now it is about time to the formal calculations. In the right-angled triangle CNM $\angle NMC = 60°$ and also in another angled triangle *SOM* $MO = SO \cdot ctg\, 60° = \frac{\sqrt{3}}{3}H$. According to the situation, the pyramid *SABC* is regular, so it is obvious that: $OC = OA = OB = 2OM = \frac{2\sqrt{3}}{3}H$, and $S_{AOB} = \frac{1}{2}OA \cdot OB \cdot \sin 120° = \frac{\sqrt{3}}{3}H^2$and $S_{SAB} = \frac{S_{AOB}}{\cos 60} = \frac{2\sqrt{3}}{3}H^2$. Taking into account the equality (*), we will finally find: $S_n = 3\sqrt{3H^2}$.

Thus, a balanced analysis of stereometric realities inside the pyramid, qualitatively made projection pattern and successfully introduced the inner projection reduced the relatively difficult problem to trivial one.

Is it possible to build in the projection picture the section of the pyramid with a content $\Sigma(CPQ)$ strongly as in the drawing model? Yes, of course, if the image height *SO* = *H*, for example, choose as original segment of the pyramid.

Graphical method. Keeping in mind that the triangle *S'O'M'* is rectangular and $\angle S'M'O' = 60°(\angle M'S'O' = 30°)$and $M'O' = \frac{1}{3}M'C'$, rotating the interval of zero level *SO* = *S'O'*(Pic. 2b) we combine the triangle *S'M'C'* with the content of image. The perpendicular *C'N'* dropped from the apex *C'* on its opposite side *S'M'* visually sets the ratio in which point *N* divides the interval *SM* (Pic. 2a) in the internal way: $\frac{S'N'}{N'M'} = \frac{SN}{NM}$.

Graphic-analytical method. In the same triangle *S'O'M'* in order to determine direction of the perpendicular *C'N'*, from the apex of the angle *O'* we drop the perpendicular *O'R'* on its hypotenuse *S'M'*. Then $\frac{(O'M')^2}{(O'S')^2} = \frac{M'R'}{R'S'} = \frac{MR}{RS} = \frac{1}{3}$. The end of the problem, grateful to the found solution, is graphically reproduced on the image content using the well-known method: $C'N' \parallel O'R'$.

Problem 3. *A right isosceles triangle is the base of the prism* $ABCA_1B_1C_1$. *Triangle's leg has length 4. The lateral edge of the prism is equal to 3. Find a degree measure of the angle and a distance between the right lines, one of which is given by point B and the middle of the leg* A_1C_1*of the triangle* $A_1B_1C_1$*and the second is an apex this right angle* A_1*and the midpoint of the hypotenuse* B_1C_1.

Obviously, the problem can be easily immediately reformulated with the emphasis on constructivism, "Build a common perpendicular for the lines, one of which passes through the middle of the point *B* and the middle of the edge A_1C_1, and the second passes through the point of $A_1$and the middle of the edge B_1C_1. Find a degree measure of the angle and distance between the right lines. Measure the original length of the common perpendicular in the image and evaluate the accuracy of the construction." This significantly will add some geometry. However, we should understand that in any way without successfully introduced imaginary of the internal projection it is too hard to identify definitely the relationship between defined and required geometric figures.

So, let BM and A_1N to be the given pair of skew lines (fig. 3). The distance between them is determined, as it is known, with the perpendicular, dropped from any point on the line A_1N on the content Σwhich is parallel to A_1N and contains *BM*. The content Σ is suitable to define using the cross section of lines *BM* and MM_1, where $MM_1 \parallel A_1N$, because this intersection is also defined with the required angle between skew lines *BM* and A_1N. In the role of an internal projection content in the direction $A \rightarrow N$ we will deliberately choose (creative moment) content Λ which is defined as a right edge of the prism. Then point *N* should be a direct projection of the right line A_1N, when the right line BM_1 should be a direct projection of the content $\Sigma(BMM_1)$, and the required joint perpendicular *PQ of the* given right lines will project on the selected projection content Λ in the life-size - as an interval, which is parallel to the height $B_1K$of the right triangle BB_1M_1, drawn from the apex of the right angle $B_1$to the hypotenuse BM_1. In this triangle $BB_1 = 3, B_1M_1 = 3\sqrt{2}(B_1C_1 = 4\sqrt{2}, B_1N = NC_1$, while MM_1 is a middle line of the triangle A_1NC_1). Therefore, the triangle BM_1M is also right ($\angle M_1 = 90°$). It has $M_1M = \frac{1}{2}A_1N = \sqrt{2}$. As a result we have: $\text{tg}\,\angle M_1\,MB = 3\sqrt{\frac{3}{2}}$ and $\angle M_1MB \approx 75°$.

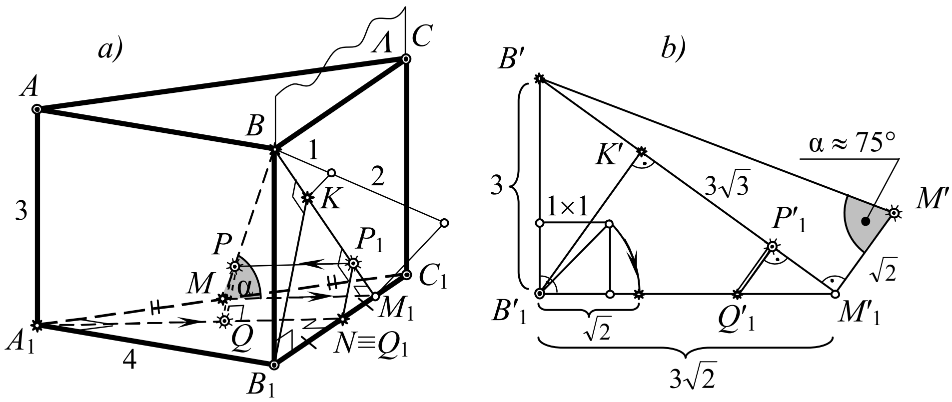

Fig. 3

The problem is solved. However, the already traversed path executed "lion's share" of constructed operations in order to quickly and efficiently complete its solution constructively. In particular, the point $K(P_1)$ on the interval BM_1 is clearly determined by the ratio $\frac{B_1M_1{}^2}{BB_1{}^2} = \frac{M_1K}{KB} = \frac{2}{1}$that simply is justified . If the point P_1 is already built, then the image of the interval *PQ* is set with the reverse projection in the direction → *A* , where $PP_1 \parallel A_1N$ and $PQ = P_1N$.

The same result is, using the well-known scheme, can also be achieved just graphically – with the combination of the image area of a right triangle BB_1M_1, choosing the prism edge as the axis of rotation$BB_1 = B'B'_1$ (see fig. 3b) and meaning, that $B^{'B'_1} = 3, B'_1M'_1 = 3\sqrt{2}, B'_1K' \perp B'M'_1$. Here the length of the interval $P'_1Q'_1 = PQ$ will have the life size.

Similar to the original one, a triangle $B'M'M'_1$can be reproduced in the best way at the so-called "carried out drawing" through the combination of its image(triangle $BMM_1$with a board (notebook) area. In the triangle $'M'M'_1\angle M'_1 = 90°$, cathetuses $M'_1M^{'} = \sqrt{2}$ and $M'_1B' = 3\sqrt{3}^{'}$, and $\angle M' \approx 75°$.

The results found graphically (upon condition of precise deletion), we propose to measure using a ruler and protractor, respectively, and compare with previous analytical (numerical) calculations. It's no secret that their precision merger will bring considerable moral satisfaction to the subject of learning and convince in the reality of first subject's facts and laws.

Today, the average school graduate does not understand the structure, is not able to clearly classify the figures of Euclidean geometry, confuses the concepts and facts, cannot use them properly in finding solutions of calculation problems of medium complexity. We are even not talking about finishing problems, and moreover about building problems. Therefore, a student is not ready for responsible, effective mastering of higher geometry.

Structural and methodological differentiation, which is clearly traced in modern science "Geometry" and not always justified choice in universities of various sections as learning ones, directly influence on the development, formation of students' skills of obtaining significant mental perception of specific concepts and facts. The image of Euclidean geometry from the school in memory of student is rest quite volume and full of actual material, significantly stingy standards of academic hours for its assimilation, unobtrusive abstract, simple and not interesting calculation problems, which are solved with the ready-made formulas and moreover, in the remaining time. As a result, the discipline seems to be unmotivated, difficult to understand and is not desired to learn.

The method of presentation and learning of Euclidean geometry should be immediately changed. Namely, not teaching formal logic and calculation components, we should scientifically substantiate, visualize and geometrize all topics of positional and metric nature and "read between the lines of geometry". Also we should recharge the visual representation of solution methods of varied, different levels problems and pick up qualified, developing problems material.

The role of professionally trained geometry teacheris in stimulating cognitive interests, intellectual development and enrichment of creative thinking instincts. The role is manifested through popularization and active involvement of latest educational technologies, advanced and scientific knowledge methods in the learning process. Teaching geometry, a true professional is able to pass a sense of harmony of geometric material to the one who learn, and show visually, in graphic form its natural beauty and eternal practical direction. He translates smartly the abstract logical inference results into the language of imaginary graphic images that return to the studying reality. This gives confidence to the practical life of the first science, its truth.

We are sure that elementary geometry occupies a special place in the teachers' training system it should be given much more attention. Moreover, we should not repeat the school program in vain, it is boring and disastrous. Teaching and learning should be performed, as a priority, in the constructive manner. It is established that structural problems in geometry are "at the top of learning"! They are rather informative and that is why they accumulate all factual material. The research proved thefact that the method of internal projection on one projection content is a graphic (graphic-analytical) method of solving stereometry calculating, building and finishing problems. A methodology of visual constructive solution on metrically defined projection drawings of a whole class of problems is mastered using this universal metric method.

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Ленчук І. Г.

Геометризація і унаочнення стереометричних задач

Пропонується в університеті вести навчання евклідової геометрії на основі її природного конструктивізму. Організувати процес системно і в повному об'ємі так, щоб в розв'язуваних задачах візуально подані побудови з осмислено уявлюваною логікою міркувань стимулювали формування професіональних компетентностей, мотивували навчально-пізнавальний інтерес. Діяльнісний, дослідницький підхід до використання закономірних понять і фактів стане базовим показником творчого розвитку особистості. Перевірена методика забезпечить становлення динамічних стереотипів уявлень і уяви, наочно-образного і логічного мислення, накопичення міцних знань геометрії в цілому і елементарної геометрії, зокрема, умінь і навичок користуватися ними в житті та роботі за обраною спеціальністю. Такий шлях суттєво підвищить рівень наукової і методичної підготовки вчителя математики. В тексті статті прикладами задач на обчислення продемонстровано можливості їх насичення істинно геометричним змістом. У руслі реалізації конструктивного підходу пропагується, як універсальний, метод внутрішнього проекціювання.

Ключові слова: геометризація, унаочнення, графічний і графоаналітичний методи, внутрішнє проекціювання, візуалізація.

Ленчук И. Г.

Геометризация и наглядное представление стереометрических задач

Предлагается в университете вести обучение евклидовой геометрии на основании её естественного конструктивизма. Организовать процесс системно и в полном объёме так, что бы в решаемых задачах визуально поданные построения с осмысленно воображаемой логикой мышления стимулировали формирование профессиональных компетентностей, мотивировали учебно-познавательный интерес. Деятельный, исследовательский подход к использованию закономерных понятий и фактов станет базовым показателем творческого развития личности. Проверенная методика обеспечит становление динамических стереотипов представлений и воображений, наглядно-образного и логического мышления, накопление прочных знаний геометрии в целом и элементарной геометрии, в частности, умений и навыков пользоваться ими в жизни и работе по избранной специальности. Такой путь существенно повысит уровень научной и методической подготовки учителя математики. В статье примерами обычных задач на вычисление продемонстрировано возможности их насыщения истинно геометрическим содержанием. В русле реализации конструктивного подхода пропагандируется, как универсальный, метод внутреннего проецирования.

Ключевые слова: геометризация, наглядное представление, графический и графоаналитический методы, внутреннее проецирование, визуализация.

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